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The quantum geometry of spin and statistics

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Abstract

Both, spin and statistics of a quantum system can be seen to arise from underlying (quantum) group symmetries. We show that the spin–statistics theorem is equivalent to a unification of these symmetries. Besides covering the Bose–Fermi case we classify the corresponding possibilities for anyonic spin and statistics. We incorporate the underlying extended concept of symmetry into quantum field theory in a generalised path integral formulation capable of handling general braid statistics. For bosons and fermions the different path integrals and Feynman rules naturally emerge without introducing Grassmann variables. We also consider the anyonic example of quons and obtain the path integral counterpart to the usual canonical approach.

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1. Introduction

While spin in quantum physics arises from the geometry of space–time, statistics is connected to the geometry of configuration space. Half-integer spin and Bose–Fermi statistics arise in three or higher dimensions, while in two dimensions more general fractional spin and anyonic statistics are possible (see [1] and references therein). In fact, both spin and statistics are related to symmetries. In the case of spin this is plainly understood in terms of the symmetries of space–time. In the case of statistics the link is more indirect. From the geometry of a configuration space of identical particles [2] one is led (in the general case) to the braid group, which acts on it by particle exchange [3]. From the representation theory of the braid group one naturally arrives at the concept of braided categories (see, e.g., [4])

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to describe statistics. While foreign to ordinary quantum field theory, such a general formulation of statistics has already been incorporated into algebraic quantum field theory [5,6]. Going further, a reconstruction theorem of quantum group theory tells us that (essentially) every braided category is the category of representations of a quantum group. Thus, for any braid statistics there is a quantum group symmetry of the theory that generates the statistics. The relevant quantum groups for anyonic statistics are known [7,4].

After reviewing these facts, we ask, in the first part of the paper (Section 2), the natural question of whether and how the (quantum) group symmetries behind spin and statistics are related. Such a connection should be expected in the Bose–Fermi case from the spin–statistics theorem [8,9]. In this case, both groups generating spin and statistics turn out to be (essentially) $\mathbb{Z}_2$. Remarkably, the statement of the spin–statistics theorem is found to be precisely equivalent to the requirement that the groups be identified. This leads to a quantum symmetry group that encodes: (a) the space–time symmetries, (b) the statistics, and (c) the spin–statistics theorem. Technically, this quantum group is the ordinary space–time symmetry (e.g., Poincaré) group as a Hopf algebra, but equipped with a non-trivial coquasitriangular structure. We proceed to explore the possible relations of spin and statistics in the more general case of fractional spin and anyonic statistics. This amounts (under certain restrictions) to a classification of all possible spin–statistics theorems which could be implemented by a unified quantum group symmetry.

It is essential for our treatment to work with quantum groups of function algebra type and not of enveloping algebra type. The global structure of the (quantum) groups, not visible in the enveloping algebra setting, is crucial in the unified description of spin and statistics. In an enveloping algebra setting one would have to provide the global information "by hand", i.e., by adjoining elements.

The second part of the paper (Section 3) addresses the question of how the description of statistics by braided categories can be incorporated into ordinary quantum field theory. This is achieved in a path integral formulation by braided quantum field theory [10], replacing and generalising the formalism of commuting versus anti-commuting variables. For Bose–Fermi statistics both the bosonic and the fermionic (Berezin) path integral as well as the different Feynman rules for bosons and fermions are recovered.

In view of the results of the first part this means that the sole input of the relevant symmetry quantum group provides a quantum field theory not only with desired space–time symmetries, but also with the right statistics and the correct spin–statistics relation. This appears to be conceptually simpler and more natural than the somewhat arbitrary introduction of anti-commuting variables for fermions.

Finally, we explore the potential of the generalised formalism by considering an example of anyonic statistics. We study the "quons" of Greenberg [11]. We translate the canonical quon relations into a braid statistics and find that braided quantum field theory provides in this case the path integral counterpart to the canonical approach.

It has been notoriously difficult to incorporate generalised statistics into canonical relations. This is exhibited in the quon case in the necessary absence of any relations between creation or annihilation operators for different momenta. This gross deviation from the usual canonical quantisation approach indicates in our opinion that perhaps canonical relations are not the right way to describe generalised statistics. Instead, the braided approach advocated here appears more flexible.

In fact, an example of a non-trivial braid statistics in the context of braided quantum field theory has been described previously [12]. This is the statistics in a quantum field theory twist-equivalent to that on the non-commutative space arising in string theory. It is not an anyonic but rather a symmetric and momentum dependent statistics.

This paper is organised as follows. Section 2 deals with the symmetries underlying spin and statistics. Sections 2.1–2.4 review the necessary foundations in a coherent fashion leading up to the main result in Section 2.5 on the unification of the symmetries of spin and statistics. Section 3 shows how braided quantum field theory implements general braid statistics into quantum field theory. Sections 3.1 and 3.2 review the braided path integral and its simplification in the bosonic case. Sections 3.3 and 3.4 show how the fermionic path integral as well as the fermionic Feynman rules are recovered from the fermionic braiding. Finally, Section 3.5 treats the anyonic example of quons.

In the following, the term quantum group is taken to mean coquasitriangular Hopf algebra. We always work over the complex numbers. For the general theory of quantum groups and braided categories we refer to Majid's book [4] and references therein.

2. The symmetries behind spin and statistics

In this section, after reviewing the emergence of spin and statistics in quantum mechanics, we discuss the formulation of statistics in terms of braided categories and quantum groups. This leads us to a unified description of space–time symmetries and statistics in terms of quantum group symmetries. A spin–statistics theorem then precisely corresponds to the identification of space–time and statistical symmetries. We describe the familiar Bose–Fermi case and classify the corresponding possibilities for anyonic spin and statistics.

2.1. Spin

We start by recalling the geometric origin of spin. In classical mechanics we require that observable quantities form a representation of the symmetry group of space–time. In quantum mechanics it is only required that such a representation is projective, i.e., it is a representation "up to a phase" [13]. However, projective representations of a Lie group are in correspondence to ordinary representations of its universal covering group [14].

Suppose, we have some connected orientable (pseudo-)Riemannian space–time manifold M. We denote its principal bundle of oriented orthonormal frames by (E, M, G), where E is the total space and G the structure group, i.e., the orientation preserving isotropy group. If M has signature (n, m) then $G = \mathrm{SO}(n, m)$. Let $\tilde{G}$ be the universal covering group of G. Denote by $(\tilde{E}, M, \tilde{G})$ the induced lift of (E, M, G) (assuming no global obstructions). [1]

[1] Strictly speaking, we should consider coverings of the global symmetry group. However, if M is a Riemannian homogeneous space, E can be identified with the global isometry group and $\tilde{E}$ with its universal cover (assuming M to be simply connected).

Given a representation of $\tilde{G}$ with label j, a field with spin j is described by a section of the corresponding bundle associated with $(\tilde{E}, M, \tilde{G})$. If j labels a representation of G itself, we say that the spin is "integer", otherwise "fractional". Consider the exact sequence

$$\pi_1(G) \hookrightarrow \tilde{G} \twoheadrightarrow G, \tag{1}$$

where $\pi_1(G)$ denotes the fundamental group of G. A representation of $\tilde{G}$ is a representation of G if and only if the induced action of $\pi_1(G)$ is trivial. Thus, loosely speaking, the "fractions" of spin are labelled by the irreducible representations of $\pi_1(G)$. In our present context (we assume at most one time direction) there arise only two different cases which we review in the following.

Let M be three-dimensional Euclidean space. Then $G = \mathrm{SO}(3)$ and the exact sequence (1) becomes

$$\mathbb{Z}_2 \hookrightarrow \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3). \tag{2}$$

With the usual conventions, irreducible representations of SU(2) are labelled by half-integers and those with an integer label descend to representations of SO(3). $\mathbb{Z}_2$ has just two inequivalent irreducible representations that distinguish between the two choices, integer or non-integer. More generally, $\pi_1(\mathrm{SO}(1,n)) = \pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for all $n \geq 3$. Thus, if the dimension of space is ≥ 3, we can only have integer and half-integer spins.

Now, let M be two-dimensional Euclidean space. We obtain the exact sequence

$$\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathrm{SO}(2). \tag{3}$$

Since the groups are Abelian, their unitary irreducible representations form themselves (Abelian) groups. In fact, these are SO(2), $\mathbb{R}$, and $\mathbb{Z}$. (The sequence (3) is Pontrjagin self-dual.) Thus, the unitary irreducible representations of $\mathbb{R}$ are labelled by real numbers and descend to SO(2) if the label is integer. The "fractional" part is labelled by $\mathrm{U}(1) = \mathrm{SO}(2)$. Since also $\pi_1(\mathrm{SO}(1,2)) = \mathbb{Z}$, we conclude that in two spatial dimensions continuous real spin is allowed.

Finally, the case of one spatial dimension is degenerate since the orientation preserving spatial isotropy group is trivial. We do not discuss this case further.

2.2. *Statistics*

In the following, we review the various possibilities for exchange statistics arising from the quantisation of a system of identical particles [2,3]. We consider particles in d-dimensional Euclidean space. Naively, the configuration space for N particles is $\mathbb{R}^{dN}$. However, due to the particles being identical, configurations which differ only by a permutation of the particle positions are to be considered identical. Furthermore, we exclude the singularities arising from the subspace $D \subset \mathbb{R}^{dN}$, where two or more particle positions coincide. Thus, the true configuration space is $(\mathbb{R}^{dN} - D)/S_N$, where S_N denotes the symmetric group acting by exchanging the particle positions. For more than one particle and more than one dimension it is multiply connected.

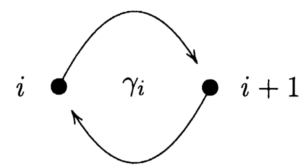


Fig. 1. Clockwise exchange of particle i with particle $i+1$.

Assuming no internal structure for the particles, quantisation can now be performed by constructing a complex line bundle with flat connection over this configuration space. The wave-function is then a section of this bundle. If we parallel transport along a non-contractible loop γ in configuration space, the wave-function picks up a phase factor $\chi(\gamma)$ coming from the holonomy of the connection. This defines a one-dimensional unitary representation of the fundamental group of the configuration space (note that this excludes parastatistics here).

For dimension $d = 2$, the fundamental group of the configuration space is the braid group on N elements, B_N. It is generated by elements $\gamma_1, \dots, \gamma_{N-1}$ with relations $\gamma_i\gamma_j = \gamma_j\gamma_i$ for $i - j \neq \pm 1$ and

$$\gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}. \tag{4}$$

To understand this more concretely, consider the inequivalent ways of exchanging two particles in the plane without coincidence. They correspond to non-contractible loops in the configuration space and thus to elements of its fundamental group. Indeed, γ_i corresponds to the exchange of particle i and particle $i+1$ in (say) clockwise direction, see Fig. 1. We represent this by a diagram which can be thought of as depicting the particle trajectories (moving from top to bottom) as they exchange, see Fig. 2. A counter-clockwise exchange corresponds to the inverse γ_i^{-1}. We can also wind the particles around each other more than once, corresponding to powers of γ_i or its inverse. Fig. 3 shows the braid relation (4) in diagrammatic notation. The one-dimensional unitary representations of the braid group are labelled by an angle θ and take the form

$$\chi(\gamma_i) = e^{i\theta} \quad \forall i. \tag{5}$$

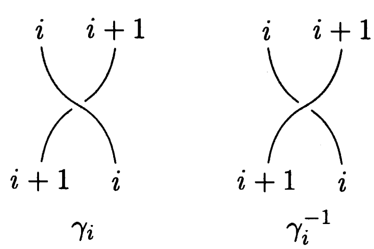


Fig. 2. Braid generators γ_i and γ_i^{-1} in diagrammatic notation.

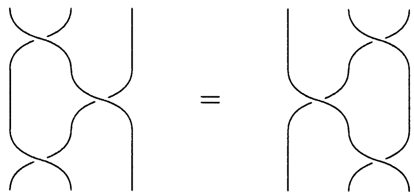


Fig. 3. Braid relation in diagrammatic notation.

This was termed θ-statistics in [3]. More generally, a statistics that is induced by representations of the braid group is called a *braid statistics*.

In dimension $d \geq 3$, the fundamental group of the configuration space is just the symmetric group S_N. It can be obtained from the braid group by imposing the extra relations $\gamma_i = \gamma_i^{-1}$. The geometric meaning of this is that the clockwise and counter-clockwise exchange of particles are equivalent (homotopic), since we can use the extra dimensions to deform one path into the other. Diagrammatically, over- and under-crossings (Fig. 2) become equivalent. The possible representations (5) reduce to just two: bosonic ($\theta = 0$) and fermionic ($\theta = \pi$) statistics.

2.3. Braided categories and statistics

In quantum field theory, multi-particle states are usually expressed in a Fock space formalism. That is, they are tensor products of one-particle states. In order to describe a general braid statistics in this context, we define invertible linear maps

$$\psi : V \otimes W \to W \otimes V$$

that exchange particles in state spaces V and W, and represent the elements γ_i of the braid group. ψ is called a *braiding*. We depict it by the same crossing diagram that we used for γ_i (Fig. 2). The diagram is now interpreted as a map, to be read from top to bottom, where the strands carry the vector spaces V and W, respectively (Fig. 4). In this formulation, we can easily express the statistics between different particles as well, by defining ψ for V and W being different spaces. Also, we are not restricted to one-dimensional representations of the braid group (or symmetric group) as was the discussion in Section 2.2. Furthermore, we can extend ψ to tensor products of multi-particle states by composing in the obvious way (Fig. 5). In fact, we can forget about the origin from the braid group or symmetric

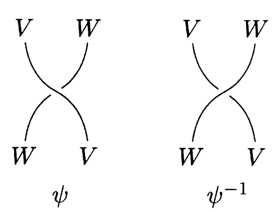


Fig. 4. The braiding and its inverse in diagrammatic notation.

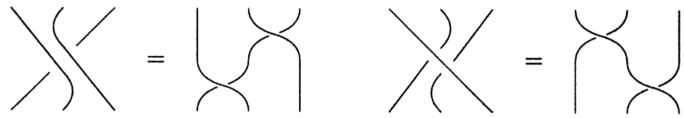


Fig. 5. Composition rules for tensor products. Close parallel strands represent tensor products.

group altogether if we implement the constraints corresponding to their relations. For the braid group this is the braid relation, expressed by the diagram in Fig. 3, which is now an identity between maps on threefold tensor products. For the symmetric group we have the additional constraint that ψ and its inverse must be identical, i.e., diagrammatically over- and under-crossings are identified.

If ψ takes the special form

$$\psi(v \otimes w) = q w \otimes v \tag{6}$$

with $q \in \mathbb{C}^*$ it is called an *anyonic statistics*, since particles obeying such statistics are called anyons [15]. If V and W are state spaces of identical particles without internal structure we recover θ-statistics (5) with $q = e^{i\theta}$ (note that we allow θ to be complex here). The general expression for Bose–Fermi statistics in this formulation is

$$\psi(v \otimes w) = (-1)^{|v| \cdot |w|} w \otimes v, \tag{7}$$

where $|v| = 0$ for bosons and $|v| = 1$ for fermions.

In fact, what we have described here is essentially what is called a *braided category* (see [16,4]). This means roughly a collection of vector spaces closed under the tensor product, together with a braiding ψ obeying the conditions depicted in Figs. 3 and 5. If $\psi = \psi^{-1}$, the braiding and the category are said to be *symmetric* (since this means restricting to representations of the symmetric group). The diagrammatics used here is the standard one for calculations in braided categories. It arises from equivalences to categories of braids, links and tangles [17,18].

A description of statistics by braided categories was first employed in the context of algebraic quantum field theory [5,6]. However, it can also be integrated into a (generalised) path integral formulation of quantum field theory [10] as will be discussed in Section 3.

2.4. *Quantum groups and statistics*

How do quantum groups come into the game? We recall a few facts about their representation theory (see [4]). As for ordinary groups, the tensor product $V \otimes W$ of representations V, W forms again a representation. However, in contrast to ordinary groups, the map $\tau : V \otimes W \to W \otimes V$ defined by $v \otimes w \mapsto w \otimes v$ is not in general an intertwiner, i.e., does not in general commute with the quantum group action. Instead, for any pair of representations, we are given a (generally non-trivial) invertible linear map $\psi : V \otimes W \to W \otimes V$, which is an intertwiner. It is encoded in the so-called *coquasitriangular structure* of the quantum group. In fact, these intertwiners precisely obey the conditions for a braiding discussed above. Thus, the category of representations of a quantum group becomes a braided category. Remarkably, the converse is also true: given a braided category, we can (under certain technical conditions) reconstruct a quantum group so that the given category arises as its category of representations. This is called Tannaka–Krein reconstruction, see [4].

Having seen that braided categories can be used to describe statistics, we can say that quantum group theory naturally integrates the notions of "spin" (representation theory) and statistics. More precisely, the reconstruction theorem tells us that for a given braid statistics (given by a braided category) there is an underlying symmetry quantum group, so that the statistics of particles is determined by their representation labels. In the following, we discuss this for anyonic statistics including the special case of Bose–Fermi statistics. The relevant quantum groups were identified by Majid [7,4] (in a dual formulation of enveloping algebras).

The quantum group generating general anyonic statistics turns out to be the ordinary group U(1), but with a non-standard coquasitriangular structure. As a quantum group it is the function algebra $\mathcal{C}(\mathrm{U}(1))$. A natural basis in terms of the coproduct are the Fourier modes $f_k : \phi \mapsto \mathrm{e}^{2\pi \mathrm{i} k\phi}$, labelled by $k \in \mathbb{Z}$. The relations are $f_k f_l = f_{k+l}$, the coproduct is $\Delta f_k = f_k \otimes f_k$, the counit is $\epsilon(f_k) = 1$, and the antipode is $S f_k = f_{-k}$. The coquasitriangular structure $\mathcal{R} : \mathcal{C}(\mathrm{U}(1)) \otimes \mathcal{C}(\mathrm{U}(1)) \to \mathbb{C}$ that generates the braiding is given by

$$\mathcal{R}(f_k \otimes f_l) = q^{kl}. \tag{8}$$

The unitary irreducible representations of U(1) are labelled by $k \in \mathbb{Z}$. The braid statistics takes the form

$$\psi(v_k \otimes v_l) = q^{kl} v_l \otimes v_k. \tag{9}$$

This reduces to expression (6) for particles in the representation $k = l = 1$.

As it will be of relevance later, we remark that the same anyonic statistics can also be generated by the group $\mathbb{R}$. As a quantum group we consider the function algebra $\mathcal{C}(\mathbb{R})$ generated by the periodic functions. The only difference to the U(1) case discussed above is that the basis $\{f_k\}$ is now labelled by real numbers $k \in \mathbb{R}$ and not just integers. Otherwise the algebraic structure is the same and (8) and (9) still hold in the same form. Representations are also labelled by $k \in \mathbb{R}$ now.

For $q = \mathrm{e}^{\mathrm{i}\theta}$ an nth root of unity we call the statistics *rational* since $\theta/2\pi$ is rational. In this case, we can restrict U(1) to the subgroup $\mathbb{Z}_n$. This corresponds in the quantum group setting to the extra relations $f_k = f_{k+n}$, so that we obtain a finite dimensional quantum group. Irreducible representations are now labelled by $k \in \mathbb{Z}_n$. However, it will be convenient for the following discussion to introduce an alternative fractional labelling by $k \in \frac{1}{n}\mathbb{Z}_n$. Expression (9) is thus modified to

$$\psi(v_k \otimes v_l) = q^{n^2 kl} v_l \otimes v_k. \tag{10}$$

Expression (6) is recovered for $k = l = \frac{1}{n}$. The special case of $\mathbb{Z}_2$ ($\theta = \pi$) is Bose–Fermi statistics (7) with $|v_0| = 0$ and $|v_{1/2}| = 1$.

The quantum groups generating anyonic statistics are summarised in Table 1 (with the special case of $\mathbb{Z}_2$ listed separately).

Table 1
Anyonic statistics generating quantum groups

Quantum group	$\mathbb{R}$	U(1)	$\mathbb{Z}_n$	$\mathbb{Z}_2$
Representation labels	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{Z}_n$	$\mathbb{Z}_2$
Statistics parameter	$q \in \mathbb{C}^*$	$q \in \mathbb{C}^*$	$q^n = 1$	$q = -1$

2.5. Unifying spin and statistics

Having found an underlying "spin" connected with statistics, the natural question arises what possible relation it can have to the geometric spin discussed in Section 2.1. This is in essence the question of what spin–statistics theorems are quantum geometrically realisable. We give a complete answer to this question in the following (under the restricting assumption that integer spin particles behave bosonic in dimension ≥ 3).

We consider first the Bose–Fermi case. Our labelling of the $\mathbb{Z}_2$ representations above by $0, \frac{1}{2}$ is already suggestive of an interpretation as the fractional part of geometric spin. The latter is also described by an $\mathbb{Z}_2$, arising from the universal covering of the isotropy group in dimensions ≥ 3. In fact, identifying the two groups is precisely equivalent to requiring the usual spin–statistics theorem to hold.

To make this more precise we consider the generic case of three-dimensional Euclidean space. Translating the exact sequence (2) into quantum group language, it takes the arrows reversed form [2]

$$\mathcal{C}(\mathrm{SO}(3)) \hookrightarrow \mathcal{C}(\mathrm{SU}(2)) \twoheadrightarrow \mathcal{C}(\mathbb{Z}_2). \tag{11}$$

Instead of inheriting the trivial coquasitriangular structure canonically associated to ordinary groups, we equip $\mathcal{C}(\mathbb{Z}_2)$ with the non-trivial coquasitriangular structure generating the Bose–Fermi statistics. This induces a non-trivial coquasitriangular structure on $\mathcal{C}(\mathrm{SU}(2))$ which precisely exhibits the usual spin–statistics relation. Explicitly, for a group-like basis $\{1, g\}$ of $\mathcal{C}(\mathbb{Z}_2)$ and a Peter–Weyl basis $\{t^l_{ij}\}$, $l \in \frac{1}{2}\mathbb{N}_0$ of $\mathcal{C}(\mathrm{SU}(2))$, the right-hand map of (11) is $t^l_{ij} \mapsto \delta_{ij} g^{2l}$. The coquasitriangular structure $\mathcal{R}(g \otimes g) = -1$ on $\mathcal{C}(\mathbb{Z}_2)$ pulls back to

$$\mathcal{R}(t^l_{ij} \otimes t^m_{kl}) = (-1)^{4lm} \delta_{ij} \delta_{kl}$$

on $\mathcal{C}(\mathrm{SU}(2))$. This induces the braiding

$$\psi(v_k \otimes v_l) = (-1)^{4kl} v_l \otimes v_k,$$

relating spin and statistics for bosons and fermions in the familiar way. The analogous construction can be made in any space–time with spatial dimension ≥ 3, since the only relevant input is that the fundamental group of the isotropy group is $\mathbb{Z}_2$. This ensures that the function algebra of the covering group is $\mathbb{Z}_2$-graded into functions that are symmetric or antisymmetric with respect to exchange of the sheets. This decomposition is also a

[2] Note that this is *not* an exact sequence of vector spaces.

decomposition into subcoalgebras. Thus, the covering group admits the coquasitriangular structure

$$\mathcal{R}(f \otimes g) = (-1)^{|f|\cdot|g|}\epsilon(f)\epsilon(g).$$

Note that we can further pull the coquasitriangular structure back to the relevant global space–time symmetry group by the same argument. For a treatment of Bose–Fermi statistics in two dimensions, see the discussion below.

We now proceed to the more complicated case of anyonic statistics. Although we can embed U(1) into SU(2), the coquasitriangular structure (8) does not pull back from $\mathcal{C}(\mathrm{U}(1))$ to $\mathcal{C}(\mathrm{SU}(2))$. The same is true for the other spin-groups. (This is easily seen by embedding through an intermediate SU(2).) Consequently, we cannot relate the statistical "spin" of anyonic statistics to geometric spin in dimension three or higher. Even in the rational case this is only possible for $q = \pm 1$, which is the Bose–Fermi case described above. Thus, we must restrict to two dimensions, where the covering of the spatial isotropy group is described by the exact sequence (3). In contrast to the Bose–Fermi case the statistical group U(1) is different from the group $\mathbb{Z}$ describing the covering. However, we can use the group $\mathbb{R}$ to generate the statistics instead (see Table 1) and identify it directly with the universal cover $\mathbb{R}$ of the isotropy group. We obtain a spin–statistics relation between anyonic statistics and continuous geometric spin. However, for $q \neq 1$, we never have the property that representations which descend to the isotropy group have bosonic statistics, i.e., trivial braiding with all other representations.

We can implement this property, however, if we only consider a finite covering of the isotropy group. This leads to the exact sequence

$$\mathbb{Z}_n \hookrightarrow \mathrm{SO}(2) \twoheadrightarrow \mathrm{SO}(2)$$

instead of (3). The spins are restricted from continuous values to nth fractions. We can now establish a spin–statistics relation by identifying the $\mathbb{Z}_n$ of the covering with the $\mathbb{Z}_n$ of rational anyonic statistics. The braiding is the one described by (10). Pullback from $\mathcal{C}(\mathbb{Z}_n)$ to $\mathcal{C}(\mathrm{SO}(2))$ corresponds to extending the representation labels from $\frac{1}{n}\mathbb{Z}_n$ to $\frac{1}{n}\mathbb{Z}$. Representations of the covering SO(2) that descend to representations of the covered SO(2) are precisely the ones that are bosonic, i.e., have trivial braiding with all other representations; $n = 2$ is the Bose–Fermi case.

Conversely, we may ask the question what possible statistics can be attached to the geometric spin, i.e., which coquasitriangular structure is admitted by the relevant (quantum) group. Its turns out that all the relevant groups are Abelian. A coquasitriangular structure on the function Hopf algebra of an Abelian group is a bicharacter on its Pontrjagin dual, i.e., its group of unitary irreducible representations. In dimension three or higher, if we require bosonic statistics for representations descending to the isotropy group, any braiding must be induced by $\mathcal{C}(\mathbb{Z}_2)$ in (11). The dual of $\mathbb{Z}_2$ is $\mathbb{Z}_2$ and there are only two bicharacters on it: the trivial one (purely bosonic statistics) and the Bose–Fermi one discussed. In two dimensions, the covering group $\mathbb{R}$ of the isotropy group is self-dual and any bicharacter corresponds to (8) for some $q \in \mathbb{C}^*$. We also see the reason now why we were not able to induce the braiding from $\mathbb{Z}$: its dual is U(1) which has only the trivial bicharacter. In two

Table 2
Possible spin–statistics relations

Spatial dimension	2	2	≥ 3
Statistics quantum group	$\mathbb{R}$	$\mathbb{Z}_n$	$\mathbb{Z}_2$
Statistics parameter	$q \in \mathbb{C}^*$	$q^n = 1$	$q = -1$
Integer spin bosonic	–	$\surd$	$\surd$

dimensions with finite covering the relevant group is $\mathbb{Z}_n$. It is self-dual and the bicharacters correspond to the different nth roots of unity leading to rational anyonic statistics. Thus, our above discussion has already exhausted the possibilities of attaching statistics to spin. Table 2 gives a summary.

Of course, even in the absence of a spin–statistics relation we can encode the space–time symmetries as well as the statistics in terms of just one symmetry (quantum) group. This is then simply the product of the two relevant (quantum) groups.

3. Braid statistics in quantum field theory

In this section, we show how general braid statistics (in the sense of Section 2) can be incorporated into quantum field theory in a path integral formulation. The framework for this is braided quantum field theory [10], a generalisation of quantum field theory to braided spaces. While the motivation for this generalisation in [10] was purely to admit quantum group symmetries, the natural interpretation of the braiding in view of Section 2 is that of particle statistics.

Indeed, we show that the correct bosonic and fermionic path integrals as well as the different Feynman rules for bosons and fermions emerge with the braiding (7) being the only input. In particular, this does not require any other a priori distinction between bosons and fermions such as, e.g., commuting versus anti-commuting variables. As an example of anyonic statistics, we study the quons of Greenberg [11], showing that our framework provides the path integral counterpart to Greenberg's canonical approach.

3.1. Braided path integral

We review the path integral of braided quantum field theory [10]. It is based on the calculus of differentiation and integration in braided categories developed by Majid [19,4] and Kempf and Majid [20].

Consider Gaussian integration on a finite [3] set of variables $\{\phi_1, \phi_2, \ldots\}$. As usual, we define the partition function [4]

$$Z := \int \mathcal{D}\phi \exp(-S(\phi)),$$

[3] To keep the discussion less abstract, we restrict it here to the finite dimensional case.
[4] The Euclidean signature of the action is chosen for definiteness and does not imply a restriction to Euclidean field theory.

and the (free) n-point functions

$$\langle \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_n} \rangle := \frac{1}{Z} \int \mathcal{D}\phi\, \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_n} \exp(-S(\phi)), \tag{12}$$

where S is a quadratic form in ϕ. We introduce differentials $\{\partial^1, \partial^2, \dots\}$ dual to the variables and impose the familiar rules

$$\int \mathcal{D}\phi\, \partial^i (\phi_{k_1}, \dots, \phi_{k_n} \exp(-S)) = 0, \tag{13}$$

$$\partial^i (\exp(-S)) = \partial^i (-S) \exp(-S). \tag{14}$$

For the general case, we assume that the variables satisfy some definite braid statistics. Recall from Section 2.3 that this means that the vector space Φ spanned by $\{\phi_1, \phi_2, \dots\}$ lives in a braided category. The statistics is encoded in the braiding $\psi : \Phi \otimes \Phi \to \Phi \otimes \Phi$, which can be thought of as representing the exchange of particles. We write more explicitly

$$\psi(\phi_i \otimes \phi_j) = \sum_{k,l} \mathsf{R}_{ij}^{kl} \phi_l \otimes \phi_k \tag{15}$$

for a matrix R. (The braid relation of Fig. 3 is now equivalent to the Yang–Baxter equation for R.) Under the braiding, the Leibniz rule for differentiation becomes itself braided. In terms of R this can be expressed by the relation

$$\partial^i \phi_j = \delta^i_j + \sum_{k,l} \mathsf{R}_{jk}^{li} \phi_l \partial^k. \tag{16}$$

The n-point functions (12) are now completely determined by the three rules (13), (14) and (16).

It was shown in [10] that a braided generalisation of Wick's theorem holds: a $2n$-point function can be expressed in terms of n 2-point functions (propagators). (An n-point function for odd n vanishes.) This is concisely expressed in the formula

$$\langle \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_{2n}} \rangle = \langle \cdot\cdot \rangle^n \circ [2n+1]!!(\phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_{2n}}), \tag{17}$$

which we explain presently. The symbol $\langle \cdot\cdot \rangle^n$ stands for n propagators, while $[2n+1]!!$ denotes a certain linear map from the $2n$-fold tensor product of Φ's into itself. Thus, the right-hand side of (17) means: take the tensor product $\phi_{k_1} \circ \cdots \circ \phi_{k_{2n}}$, apply the map $[2n+1]!!$, then insert the result into n propagators. The map $[2n+1]!!$ is a composition

$$[2n-1]!! := ([1] \otimes \mathrm{id}^{2n-1}) \circ ([3] \otimes \mathrm{id}^{2n-3}) \circ \cdots \circ ([2n-1] \otimes \mathrm{id}), \tag{18}$$

called the *braided double factorial*. It is built out of *braided integers*, which are linear maps defined in terms of the braiding ψ as

$$[m] := \mathrm{id}^m + \mathrm{id}^{m-2} \otimes \psi^{-1} + \cdots + \psi^{-1}_{1,m-1}. \tag{19}$$

The diagrammatic interpretation introduced in Section 2 makes this more transparent. We represent the braided integer $[m]$ by the linear combination of diagrams depicted in Fig. 6.

Fig. 6. Braided integer.

Each summand of (19) is represented by one diagram, containing m strands. As a map, it is to be read from top to bottom. Each strand corresponds to one tensor factor of Φ (i.e., one variable). Crossings correspond to the braiding ψ or its inverse ψ^{-1} (Fig. 4), while lines that do not cross simply represent the identity map on that tensor factor. The composition of maps as in (18) is expressed in terms of diagrams by gluing the strands together, one diagram on top of the other. See for example Fig. 7, representing the braided double factorial [5]!!. Sums of diagrams are composed by summing over all compositions of individual diagrams. One can think of the diagrams as representing the paths of an ensemble of particles. Each strand is then the track of a particle moving from top to bottom with crossings corresponding to exchanges. Alternatively, if one represents the application of the propagators by connecting the corresponding strands at the bottom one recovers precisely the usual pictures drawn to illustrate the ordinary Wick theorem. However, the pictures obtained here carry additional information encoded in the type of crossing.

We stress that in contrast to ordinary (path) integrals there are no algebra relations between the ϕ's. The space Φ just generates a free (non-commutative) algebra. However, it is possible to impose relations anyway, provided they are compatible with the braided differentiation. (More precisely, the relations must be braided coalgebra maps for the primitive coproduct on Φ.) Such relations commute with the braided Wick theorem in the sense that imposing the relations first and then evaluating (17) is the same as evaluating (17) first and then imposing the relations.

We restrict now to the case where the braiding simply permutes variables with an extra factor. That is, we assume $\mathsf{R}_{ij}^{kl} = \alpha_{ij}\delta_i^k\delta_j^l$ in (15). Explicitly,

$$\psi(\phi_i \otimes \phi_j) = \alpha_{ij}\phi_j \otimes \phi_i. \tag{20}$$

This is sufficient for considering bosonic, fermionic, and anyonic statistics. The braided integers become sums of permutations equipped with extra factors. Consequently, we can

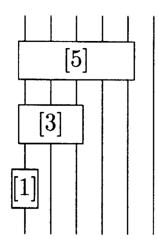


Fig. 7. The braided double factorial [5]!!.

express the braided Wick theorem (17) in a more familiar way:

$$\langle\phi_{k_1},\phi_{k_2},\dots,\phi_{k_{2n}}\rangle=\sum_{\text{pairings}}\alpha(P)\langle\phi_{k_{P_1}}\phi_{k_{P_2}}\rangle\cdots\langle\phi_{k_{P_{2n-1}}}\phi_{k_{P_{2n}}}\rangle, \tag{21}$$

where the sum runs over all permutations P of $\{1,\dots,2n\}$ leading to inequivalent pairings. Generically, α is some complicated function of P, built out of the α_{ij}. It does not, in general, define a representation of the symmetric group. Note also that the order of the variables in each propagator on the right-hand side is relevant. It is such that the two variables are in the same order on both sides of the equation.

3.2. Bosonic path integral

Setting $\alpha_{ij}=1$ in (20) defines bosonic statistics. In this case, the braided path integral and the Feynman rules reduce by construction to the ordinary ones of bosonic quantum field theory [10]. Nevertheless, we include the bosonic integral here for completeness and to prepare the ground for the fermionic case.

The Leibniz rule (16) becomes the ordinary one $[\partial^i,\phi_j]=\delta^i_j$ and we recover the relevant bosonic differentiation and integration rules. In expression (21), we get $\alpha(P)=1$ and arrive at the bosonic Wick theorem, which merely expresses the combinatorics of grouping variables into pairs. Writing $S(\phi)=\frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j$ for a symmetric matrix A, the resulting propagator is $\langle\phi_k\phi_l\rangle=A^{-1}_{kl}$.

As a combinatorial exercise we can count the number of terms in (21) by giving each propagator the numerical value 1. This amounts to replacing the braided integers in (17) by ordinary integers. The braided double factorial turns into an ordinary double factorial and we obtain the value $(2n-1)!!=(2n)!/(n!2^n)$, which is precisely the number of ways in which we can arrange $2n$ variables into pairs of two.

For the case of conjugated variables $\{\phi_1,\phi_2,\dots\}$ and $\{\bar\phi_1,\bar\phi_2,\dots\}$, we require S to have the form $S(\phi)=\sum_{i,j}\bar\phi_i B_{ij}\phi_j$ for a matrix B. This is the same, however, as taking all variables $\{\phi_1,\phi_2,\dots,\bar\phi_1,\bar\phi_2,\dots\}$ together and requiring A to have the form

$$A=\begin{pmatrix}0 & B^{\mathrm{T}}\\ B & 0\end{pmatrix}.$$

The propagator becomes $\langle\phi_k\bar\phi_l\rangle=\langle\bar\phi_l\phi_k\rangle=B^{-1}_{kl}$ with propagators of two un-barred or two barred variables vanishing. Consequently, Wick's theorem specialises to its familiar form for conjugated variables

$$\langle\phi_{k_1}\bar\phi_{l_1}\cdots\phi_{k_n}\bar\phi_{l_n}\rangle=\sum_{\text{permutations }P}\langle\phi_{k_1}\bar\phi_{l_{P_1}}\rangle\cdots\langle\phi_{k_n}\bar\phi_{l_{P_n}}\rangle, \tag{22}$$

where the sum runs over all permutations P of $\{1,\dots,n\}$.

3.3. Fermionic path integral

Setting $\alpha_{ij} = -1$ in (20) defines fermionic statistics. We show that the resulting path integral is equivalent to the Berezin path integral for fermions in standard quantum field theory.

The Leibniz rule (16) becomes $\{\partial^i, \phi_j\} = \delta^i_j$. This is indeed the familiar expression for Grassmann variables, which are usually employed to perform the fermionic integration. Furthermore, the other rules (13) and (14) that we have required to define n-point functions turn out to hold also for Grassmann variables. This is quite obvious for (13), since differentiation and integration are the same for Grassmann variables and differentiating twice by the same variable must result in zero. Writing $S(\phi) = \frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j$ with anti-symmetric matrix A, (14) follows from the observation that the relations $[\partial^i, S] = A_{ij}\phi_j$ and $[\phi_i, S] = 0$ have the same commutator form as in the bosonic case, since S is quadratic. Thus, fermionic braided integration and integration with Grassmann variables must agree. Indeed, in expression (21) $\alpha(P)$ becomes the signature of the permutation P. This is Wick's theorem for fermions. Also, the propagator becomes the correct one $\langle\phi_k\phi_l\rangle = A^{-1}_{kl}$.

As in the bosonic case, we can play the game to assign each propagator the numerical value 1. This time we count the difference between the number of terms contributing with a plus sign and those with a minus sign in (21). In the diagrammatic language, this amounts to replacing any diagram by 1 or -1 depending on whether it contains an even or odd number of crossings. For the braided integers this means that $[m]$ takes the value 1 if m is odd and zero if m is even. Since the braided factorial is a product of odd integers it takes the value 1, which is the desired result.

For conjugated variables $\{\phi_1, \phi_2, \ldots\}$ and $\{\bar{\phi}_1, \bar{\phi}_2, \ldots\}$, we write $S(\phi) = \sum_{i,j}\bar{\phi}_i B_{ij}\phi_j$ for a matrix B. This is the same as requiring A to have the form

$$A = \begin{pmatrix} 0 & -B^{\mathrm{T}} \\ B & 0 \end{pmatrix}. \tag{23}$$

The propagator becomes $\langle\phi_k\bar{\phi}_l\rangle = -\langle\bar{\phi}_l\phi_k\rangle = B^{-1}_{kl}$ with propagators of two un-barred or two barred variables vanishing. Similar to the bosonic case, Wick's theorem specialises to the form (22), although with a factor inserted that takes the value 1 or -1 depending on the signature of the permutation P.

3.4. Fermionic Feynman diagrams

In braided quantum field theory, no a priori distinction is made in the treatment of fields with different statistics in (generalised) Feynman diagrams. While for bosonic fields the correct Feynman rules are obtained by construction, this is less apparent for fermionic fields. We show in the following that the correct fermionic Feynman rules indeed emerge.

We briefly recall the rules for composing braided Feynman diagrams, i.e., the generalised Feynman diagrams of braided quantum field theory. This extends the diagrammatic language already introduced. As usual, diagrams are composed of propagators and vertices. However,

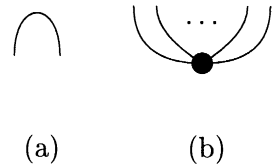


Fig. 8. Propagator (a) and vertex (b).

Fig. 9. Decomposition of the fermionic propagator.

they have to be arranged in a certain way. Propagators, which are represented by arches (Fig. 8a), are all drawn next to each other at the top of a diagram. Vertices, represented by dots that collect several strands together (Fig. 8b) are drawn next to each other at the bottom. The propagators are connected with the vertices by lines which can cross. These crossings are precisely the braiding or its inverse (Fig. 4). External lines simply end on the bottom line of the diagram without meeting a vertex.

For bosons, all crossings are trivial and we recover the usual bosonic Feynman rules. For fermions, over- and under-crossings are still identical, but introduce a factor of -1. This is the only difference between bosons and fermions in braided Feynman diagrams. At first sight this appears to be at odds with standard quantum field theory, which prescribes no factor for line crossings, but introduces extra rules for fermions instead: (a) each exchange of external fermion lines introduces a factor of -1, (b) each internal fermion loop contributes a factor of -1. In fact, both prescriptions are equivalent as we proceed to show in the following.

It is easy to see how rule (a) comes about. Exchanging external fermion lines is achieved by introducing (or removing) crossings. The number of crossings is necessarily odd, since the exchanged lines cross once, while any other lines are crossed twice (once by each of the two which are to be exchanged). Furthermore, crossings of external lines with loops or of loops with loops do not contribute since they always appear in pairs. It remains to be shown how rule (b) arises.

First, we note that ordinary fermions are described by conjugated variables. Thus, the propagator consists of two components corresponding to the two blocks in (23). Usually, one picks out one component and indicates which one it is by an arrow (Fig. 9). The two components have a relative minus sign as in (23) due to fermionic anti-symmetry. The same applies to fermion vertices (Fig. 10).[5] Since only matching orientations contribute, each fermion line decomposes into two components with consistently chosen orientation of propagators and vertices.

[5] Note that the relative choice of positive orientation between propagator and vertex is the choice of sign for the vertex term in the action.

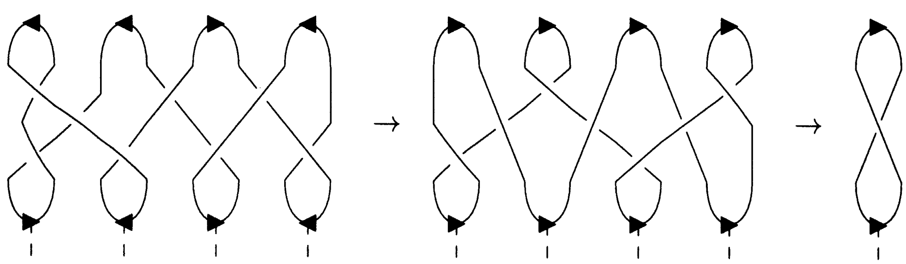
Fig. 10. Decomposition of fermionic vertices. The dotted lines (drawn downwards for ease of notation) represent other fields.

Consider now a fermion loop (see the example in Fig. 11a for illustration). We have to sum over both orientations of the loop in general. We consider the contribution of one of the two orientations. Its sign is determined from the various crossings and orientation choices of the propagators and vertices. To simplify, we choose the positive orientation and twist any propagators and vertices with the negative orientation around (Fig. 11b). This does not alter the sign, since crossings and orientation changes are introduced in equal number. Now, the sign contribution of the diagram is determined by the number of crossings modulo 2. To find it, we remove propagators and vertices pair-wise by straightening out lines, keeping in mind that we are always allowed to change over- in under-crossings and vice versa. This removes crossings only pair-wise and leaves the sign invariant. We are left with just one propagator and vertex (Fig. 11c). This diagram must have one (or an odd number of) crossing. Thus, the overall factor is -1, in agreement with standard quantum field theory.

As a further remark, the strict rules for the arrangement of propagators and vertices in braided Feynman diagrams can be relaxed to the ordinary rules if the braiding is symmetric ($\psi = \psi^{-1}$), as is the case for fermions. This is discussed in [12].

3.5. Anyonic statistics and quons

In this section, we investigate anyonic statistics, i.e., we are interested in the case $\alpha_{ij} = q$ for $q \in \mathbb{C}^*$ in (20). Since there is no standard quantum field theory of anyons to compare with, we start from a canonical approach. This also sheds new light on the bosonic and fermionic case from this point of view. More specifically, we consider the "quons" which provide an interesting example of anyons studied by Greenberg [11].



(a) (b) (c)

Fig. 11. Evaluating the sign of a fermion loop.

Consider the relations

$$a_k a_l^\dagger = \delta_{kl} + q a_l^\dagger a_k \tag{24}$$

between creation and annihilation operators. Greenberg's treatment of this algebra is motivated by the possibility of small violations of bosonic ($q = 1$) or fermionic ($q = -1$) statistics. However, we need not take this point of view here.

In contrast to the ordinary canonical approach, no relations among a's or $a^\dagger$'s are introduced. In fact, such relations are not needed for normal ordering or the calculation of vacuum expectation values, as was stressed in [11]. We are going one step further by remarking that relation (24) is only ever evaluated in one direction: from left to right. Thus, one could interpret (24) as defining the exchange statistics between a particle ($a_l^\dagger$) and a "hole" (a_k), where the δ-term just comes from the operator picture, analogous to expression (16). The corresponding braiding is

$$\psi^{-1}(a_k \otimes a_l^\dagger) = q a_l^\dagger \otimes a_k. \tag{25}$$

The choice of ψ^{-1} over ψ is to conform with the conventions of braided quantum field theory, where only ψ^{-1} appears in (18). In fact, we wish to make the whole Hilbert space of states into a braided space, in the spirit of Section 2. In order for expressions with an equal number of creation and annihilation operators ("zero particle number") to behave bosonic, we need to impose

$$\psi^{-1}(a_k \otimes a_l) = q^{-1} a_l \otimes a_k, \qquad \psi^{-1}(a_k^\dagger \otimes a_l^\dagger) = q^{-1} a_l^\dagger \otimes a_k^\dagger. \tag{26}$$

The particles and holes obey anyonic statistics among themselves.

We take the statistics generating group according to Table 1 to be U(1). Thus, we have the general expression

$$\psi(v \otimes w) = q^{|v|\cdot|w|} w \otimes v \tag{27}$$

for the statistics. Particles are in the representation $|a_k^\dagger| = 1$ and holes in the representation $|a_k| = -1$, so that we recover (25) and (26).

A massive real scalar field is expressed in terms of creation and annihilation operators as

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2\omega_k}} (a_k\, \mathrm{e}^{-\mathrm{i}k\cdot x} + a_k^\dagger\, \mathrm{e}^{\mathrm{i}k\cdot x})$$

with $\omega_k = \sqrt{k^2 + m^2}$. We split it as usual into the components $\phi(x) = \phi^+(x) + \phi^-(x)$, where $\phi^+(x)$ only contains annihilation operators, while $\phi^-(x)$ only contains creation operators. We can view this formally as a decomposition of the space of classical fields $\Phi = \Phi^+ \oplus \Phi^-$. The statistics inherited from the canonical picture is given by the U(1) representation labels $|\phi^+(x)| = -1$ and $|\phi^-(x)| = 1$. As a remark, we observe that upon reducing U(1) to $\mathbb{Z}_2$, we have $1 \cong -1$ as representations. This can be seen to be the reason why no analogous splitting of the field was necessary in the fermionic case.

With the braiding defined on the classical field, the path integral description of the quon is now precisely given by the path integral of braided quantum field theory. Indeed, the

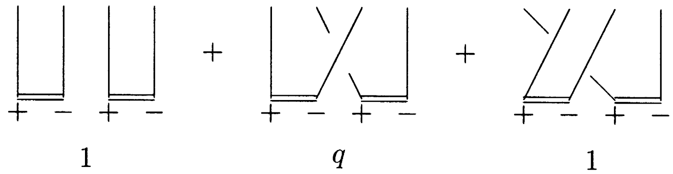


Fig. 12. The contributions to the quon 4-point function.

braided Wick theorem (17) specialises in this case to the one found by Greenberg [11]. We consider the example of the free 4-point function. Its decomposition into propagators is given by

$$\begin{aligned}&\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\\&\quad=\langle\phi(x_1)\phi(x_2)\rangle\langle\phi(x_3)\phi(x_4)\rangle+q\langle\phi(x_1)\phi(x_3)\rangle\langle\phi(x_2)\phi(x_4)\rangle\\&\qquad+\langle\phi(x_2)\phi(x_3)\rangle\langle\phi(x_1)\phi(x_4)\rangle.\end{aligned}\tag{28}$$

This reproduces (37)–(39) in [11]. [6]

To see how (28) comes about consider Fig. 12. The braided double factorial $[3]!!=[3]\otimes\mathrm{id}$ produces a sum of three diagrams. At the bottom we have indicated by horizontal double lines the evaluation by the propagators. In order to see what factors the braidings introduce we note that only the combination $\langle\phi^+(x)\phi^-(y)\rangle$ makes a contribution to the propagator. Accordingly, we have written below each strand the sign indicating the relevant field component carried by the strand. The evaluation is now simply determined by the statistics of the relevant field components: a braiding of a $+$ with a $-$ field gives a factor of q, while braidings among $+$ or $-$ fields give a factor of q^{-1}. In this way, any free n-point function can be easily evaluated. Note that the rule for obtaining the q-factors given by Greenberg appears to be slightly different, but is equivalent. If, while fixing the attachments of the strands at the top line we deform the strands (with the attached propagators) so as to minimise the number of intersections, we are only left with intersections between fields with different sign labels. These all introduce factors of q. This is Greenberg's description.

Finally, we consider the issue of the statistics of bound states of quons. It was found in [21] that bound states of n quons have a statistics parameter of q^{n^2}. In fact, in our formulation this follows from the knowledge of the (quantum) group symmetry behind the statistics. A quon and its creation operator is in a 1-representation of the statistics generating U(1). A quon hole and the annihilation operator are in the -1-representation. Thus, an n-quon state or operator that increases the quon number by n lives in an n-representation. By formula (27), we find that the statistics factor between two such objects is $q^{n\cdot n}$.

Although, usually considered in the context of small violations of Bose or Fermi statistics in higher dimensions, our analysis suggests that it would be quite natural to consider quons in

[6] Greenberg uses a complex scalar field. However, it is clear how to obtain (37)–(39) in [11] from (28): just insert the †'s and remove propagators that are not pairs of a ϕ and a $\phi^\dagger$.

two dimensions where a spin–statistics relation can be established quantum-geometrically as shown in Section 2.5.

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